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We consider the vacuum energy for a scalar field subject to a frequency dependent boundary condition. The effect of a frequency cut-off is described in terms of an *incomplete*  $\zeta$ -function. The use of the Debye asymptotic expansion for Bessel functions allows to determine the dominant (volume, area, ...) terms in the Casimir energy. The possible interest of this kind of models for dielectric media (and its application to sonoluminescence) is also discussed.

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## I. INTRODUCTION

The Casimir effect [1,2] arises as a distortion of the vacuum energy of quantized fields due to the presence of boundaries (or nontrivial topologies) in the quantization domain. This effect, which has a quantum nature associated with the zero-point oscillations in the vacuum state, is significant in diverse areas of physics, from statistical physics to elementary particle physics and cosmology.

In particular, in the last years there has been a great interest in the Casimir energy of electromagnetic fields in the presence of dielectric media, due to Schwinger's suggestion [3] that it could play a role in the explanation of the phenomenon of sonoluminescence [4].

The results obtained on this subject by different groups through several calculation techniques (as Green's functions methods, van der Waals forces,  $\zeta$ -function methods and asymptotic developments for the density of states - see references [5–14] among others) are rather controversial, and some basic issues remain to be clarified. In this respect, it is our aim to contribute to the understanding of the problem by studying a simplified model, which incorporates a frequency cut-off in the boundary conditions at the separation between media, to emulate the behavior of real dielectrics.

In what follows we consider a simple model of a scalar field subject to frequency dependent local boundary conditions on the surface of a sphere. Thus our main goal is to establish a method for calculating the change of the Casimir energy of the field when the radius of the sphere is varied, in a situation where the boundary conditions impose a *physical* frequency cut-off  $\Omega$ .

To this end we consider the very simple case of a scalar field whose modes corresponding to eigenfrequencies  $\omega \leq \Omega$  are confined to the interior of a sphere of radius  $R$ , satisfying local homogeneous boundary conditions.

On the other hand, we will assume that the boundary is completely transparent for those modes with  $\omega > \Omega$ . Therefore, their contribution to the difference of Casimir energies for two different values of  $R$  will cancel out, no matter the regularization employed for its definition. Consequently, we will subtract these contributions, which amounts to a redefinition of the reference energy level in an  $R$ -independent way.

For the evaluation of the vacuum energy of the low frequency modes we will employ asymptotic expansions in an *incomplete*- $\zeta$  summation technique, to be discussed in the following. This approach will allow for the identification of the *volume*, *surface*, ..., terms in the Casimir energy.

This method will be applied in a forthcoming paper [15] to a similar model for the case of the electromagnetic field in the presence of dielectric media. This could be of interest to investigate the role the Casimir energy can play in explaining the phenomenon of the sonoluminescence [4].

## II. THE MODEL AND ITS INCOMPLETE $\zeta$ -FUNCTION

Let us consider a free scalar field in  $\mathbf{R}^3$  satisfying at the surface of a sphere of radius  $R$ , local boundary conditions which depend on the frequency  $\omega$  of the field modes.

We will make the assumption that the boundary is completely transparent for the modes of frequencies greater than a cut-off  $\Omega$ , while for  $\omega \leq \Omega$  the modes satisfy Dirichlet boundary conditions,

$$\left(\Delta + \frac{\omega^2}{c^2}\right)\psi_\omega(\vec{r}) = 0, \text{ for } r < R, \quad (1)$$

$$\psi_\omega(\vec{r})|_{r=R} = 0,$$

being confined to the interior of the sphere.

Writing  $\psi_\omega(\vec{r}) = f_l(r)Y_l^m(\theta, \varphi)$ , we get for the radial function

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{\omega^2}{c^2}\right)f_l(r) = 0, \text{ for } r < R, \quad (2)$$

where the eigenfrequencies are determined by imposing the condition

$$f_l(r)|_{r=R} = 0. \quad (3)$$

The solutions regular at the origin are given by  $f_l(r) = \sqrt{\frac{\pi}{2z}} J_\nu(z)$ , with  $\nu = l + 1/2$  and  $z = \omega_{\nu,n} r/c$ , where the eigenfrequencies are

$$\omega_{\nu,n} = \frac{c}{R} j_{\nu,n}, \quad (4)$$

being  $j_{\nu,n}$  the  $n$ -th zero of the Bessel function  $J_\nu(z)$ .

We will be interested in the difference between the vacuum energies of two situations differing in the value of  $R$ . Then, we can disregard the contributions of those modes with frequencies  $\omega > \Omega$  because, being independent of the position of the boundary, their contributions cancel out (whatever the regularization employed in defining the vacuum energy would be). This simply amounts to an  $R$ -independent subtraction, which is nothing but a redefinition of the zero energy level.

Therefore, we should evaluate the (finite) sum

$$E(R) = \sum_{\nu=1/2}^{\nu_0} 2\nu \sum_{n=1}^{N_\nu} \frac{1}{2} \hbar \omega_{\nu,n}, \quad (5)$$

where  $N_\nu$  is the number of positive zeroes of  $J_\nu(z)$  less than or equal to  $x = \Omega R/c$ , the factor  $2\nu = 2l + 1$  is the eigenvalue degeneracy, and  $\nu_0$  is the maximum value of  $\nu$  for which  $N_\nu \geq 1$ .

We are interested in an analytic, rather than numeric, evaluation of eq. (5). Although this is a finite sum, we will employ a summation method based on the evaluation of an *incomplete*  $\zeta$ -function, an approach which could be applied in more complex situations. We can employ the following representation:

$$\sum_{n=1}^{N_\nu} j_{\nu,n} = \sum_{n=1}^{N_\nu} j_{\nu,n}^{-s} \Big|_{s=-1}, \quad (6)$$

where the sum in the right hand side obviously exists for any  $s \in \mathbf{C}^1$ .

Since  $J_\nu(z)$ , for  $\nu > -1$ , has only real zeros, and its non-vanishing zeros are all simple [16], we can employ the Cauchy theorem to represent the sum in the r.h.s. of (6) as an integral on the complex plane,

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<sup>1</sup>Notice that the sum in the r.h.s. of eq. (6) evaluated at  $s = 0$  gives  $N_\nu$ , the number of eigenfrequencies contributing to the Casimir energy of the field (after the subtraction made to define it) for a given value of the angular momentum  $l = \nu - 1/2$ .

$$\sum_{n=1}^{N_\nu} j_{\nu,n}^{-s} = \frac{1}{2\pi i} \oint_C z^{-s} \frac{J'_\nu(z)}{J_\nu(z)} dz, \quad (7)$$

where the curve  $C$  encircles counterclockwise the  $N_\nu$  first positive zeros of  $J_\nu(z)$ .

For  $\Re(s)$  large enough, the contour  $C$  can be deformed into two straight vertical lines, one crossing the horizontal axis at  $\Re(z) = x$  and the other at  $\Re(z) = 0^+$ . Indeed, expressing the integrand in terms of the modified Bessel function [17]

$$I_\nu(w) = e^{-i\frac{\pi}{2}\nu} J_\nu(e^{i\frac{\pi}{2}} w) \quad (8)$$

(valid for  $-\pi < \arg(w) \leq \pi/2$ ), and taking into account its asymptotic behavior for large arguments [17], it is easily seen that, for  $0 < x \neq j_{\nu,n}, \forall n$ , the integral

$$\zeta_\nu(s, x) \equiv \frac{-1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{-s} \frac{J'_\nu(z)}{J_\nu(z)} dz, \quad (9)$$

converges absolutely and uniformly in the open half-line  $s > 1$ , from which it can be meromorphically extended to the whole complex  $s$ -plane.

Therefore, for  $s > 1$ ,

$$\sum_{n=1}^{N_\nu} j_{\nu,n}^{-s} = \zeta_\nu(s, 0^+) - \zeta_\nu(s, x). \quad (10)$$

And, since the left hand side of (10) is holomorphic in  $s$ , the singularities of  $\zeta_\nu(s, x)$  must be independent of  $x$ .

On the other hand, for  $y > 0$  [17],

$$I_\nu(-y - ix) = e^{-i\pi\nu} I_\nu(y + ix) \quad (11)$$

$$I_\nu(y + ix) = (I_\nu(y - ix))^*.$$

So, for real  $s > 1$  we can write

$$\zeta_\nu(s, x) =$$

$$\Re \left\{ \frac{-x^{1-s}}{\pi} e^{-i\frac{\pi}{2}(s+1)} \int_0^\infty (y-i)^{-s} \frac{I'_\nu(x(y-i))}{I_\nu(x(y-i))} dy \right\}. \quad (12)$$

In order to construct the analytic extension of  $\zeta_\nu(s, x)$  to  $s \simeq -1$ , we subtract and add to the integrand in (12) the first few terms obtained from the uniform asymptotic (Debye) expansion [17] of the Bessel functions,

$$\frac{I'_\nu(\nu t)}{I_\nu(\nu t)} = \frac{1}{\nu} D_\nu(t) + \mathcal{O}(\nu^{-3}), \quad (13)$$

where

$$D_\nu(t) = \nu D^{(1)}(t) + D^{(0)}(t) + \nu^{-1} D^{(-1)}(t) = \frac{\nu \sqrt{1+t^2}}{t} - \frac{t}{2(1+t^2)} + \frac{4t-t^3}{8\nu(1+t^2)^{\frac{5}{2}}}. \quad (14)$$

valid for large  $\nu$  with fixed  $t$ . We will see that this approximation allows for the identification of the volume, surface, ... contributions to the vacuum energy.

Changing the integration variable in eq. (12) to  $t \equiv z(y - i)$ , with  $z = x/\nu > 0$ , we get

$$\zeta_\nu(s, x) = \Re \left\{ \frac{-\nu^{-s}}{\pi} e^{-i\frac{\pi}{2}(s+1)} \int_{-iz}^{\infty-iz} t^{-s} \frac{d(\ln I_\nu(\nu t))}{dt} dt \right\}. \quad (15)$$

So, we must consider the integral

$$\begin{aligned} \int_{-iz}^{\infty-iz} t^{-s} \frac{d(\ln I_\nu(\nu t))}{dt} dt &= \int_{-iz}^{\infty-iz} t^{-s} D_\nu(t) dt + \\ &+ \int_{-iz}^{\infty-iz} t^{-s} \left\{ \frac{d(\ln I_\nu(\nu t))}{dt} - D_\nu(t) \right\} dt. \end{aligned} \quad (16)$$

The second integral in the right hand side of this equation converges for  $s > -2$ , since the integrand can be estimated by means of the next ( $\mathcal{O}(\nu^{-3})$ ) term in the Debye expansion (eq. (13)), which behaves as  $\mathcal{O}(t^{-3})$  for large  $|t|$ . It can be numerically evaluated at  $s = -1$ . This will not be done in this paper.

In the following, we will consider only the first integral in the right hand side of (16), retaining the first few terms of its expansion in powers of  $\nu^{-1}$  consistent with the approximation made in eq. (13).

Notice that the integrand is an algebraic function, having singularities at  $t = 0, \pm i$ , and behaving as  $\mathcal{O}(t^0)$  for large  $|t|$ . This integral converges absolutely and uniformly for  $s > 1$ , where it defines an analytic function to be meromorphically extended to the region of interest of the parameter  $s$ . As we will see, this extension reveals the singularities of  $\zeta(s, x)$  as simple poles, whose residues are independent of  $x$  (a necessary condition to give a finite result in (10) for any  $s$ ).

In fact, by virtue of the analyticity of the integrand, we can change the path of integration to write

$$\int_{-iz}^{\infty-iz} t^{-s} D_\nu(t) dt =$$

$$\begin{aligned} &\int_{-iz}^1 t^{-s} \left( \frac{\nu \sqrt{1+t^2}}{t} - \frac{t}{2(1+t^2)} + \frac{4t-t^3}{8\nu(1+t^2)^{\frac{5}{2}}} \right) dt \\ &+ \int_1^\infty t^{-s} \left\{ \frac{\nu \sqrt{1+t^2}}{t} - \frac{t}{2(1+t^2)} + \frac{4t-t^3}{8\nu(1+t^2)^{\frac{5}{2}}} - \right. \\ &\quad \left. - \left[ \nu \left( 1 + \frac{1}{2t^2} \right) - \frac{1}{2t} - \frac{1}{8\nu t^2} \right] \right\} dt + \\ &+ \int_1^\infty t^{-s} \left[ \nu \left( 1 + \frac{1}{2t^2} \right) - \frac{1}{2t} - \frac{1}{8\nu t^2} \right] dt. \end{aligned} \quad (17)$$

The first integral in the r.h.s. of eq. (17), containing the whole dependence on  $x = \nu z$ , is holomorphic in  $s$  and can be directly evaluated at the required value of this parameter. On the half-line  $(1, \infty)$  we have subtracted and added the first terms in the series expansion of  $D_\nu(t)$  for large  $t$ , thus making the second integral to converge for  $s > -2$ . The third one must be evaluated for  $s > 1$  and then analytically continued to the relevant values of  $s$ . This can be exactly done, its contribution to  $\zeta_\nu(s, x)$  in eq. (15) being the real part of

$$\frac{e^{-\frac{i}{2}\pi(1+s)}}{\nu^{1+s} 8\pi} \left( \frac{8\nu^2}{1-s} + \frac{4\nu}{s} + \frac{1-4\nu^2}{1+s} \right) \quad (18)$$

This expression has simple poles at  $s = 0, \pm 1$ , which are the only singularities of  $\zeta_\nu(s, x)$  for  $\Re(s) > -2$ . Notice that the residues of  $\zeta_\nu(s, x)$  are independent of  $x$ ,

$$\begin{aligned} \text{Res } \zeta_\nu(s, x)|_{s=1} &= \frac{1}{\pi}, \\ \text{Res } \zeta_\nu(s, x)|_{s=0} &= 0, \\ \text{Res } \zeta_\nu(s, x)|_{s=-1} &= \frac{1-4\nu^2}{8\pi}, \end{aligned} \quad (19)$$

and in agreement with the results in [18] (where  $\zeta_\nu(s, 0^+)$  is studied).

For example, for  $\zeta_\nu(s, x)$  around  $s = -1$  and for  $\nu < x$  (which will be needed in Section IV to evaluate the vacuum energy), one straightforwardly obtains the Laurent expansion

$$\begin{aligned}
\zeta_\nu(s, x > \nu) &= \frac{1 - 4\nu^2}{8\pi(1+s)} + \\
&+ \left[ \frac{(4\nu^2 - 1)}{8\pi} \left[ \log\left(\frac{\nu}{2}\right) + \log\left(z + \sqrt{z^2 - 1}\right) \right] - \frac{\nu^2}{4\pi} + \right. \\
&\left. - \frac{\nu^2}{2\pi} z \sqrt{z^2 - 1} - \frac{3z - 8z^3}{24\pi(z^2 - 1)^{3/2}} - \frac{1}{3\pi} + \mathcal{O}(\nu^{-1}) \right] + \\
&+ \mathcal{O}(s + 1),
\end{aligned} \tag{20}$$

with fixed  $z = x/\nu \gtrsim 1$ .

On the other hand, for  $x \rightarrow 0^+$  a similar calculation leads to

$$\begin{aligned}
\zeta_\nu(s, x = 0^+) &= \frac{1 - 4\nu^2}{8\pi(1+s)} + \left[ \frac{(4\nu^2 - 1)}{8\pi} \log\left(\frac{\nu}{2}\right) - \right. \\
&\left. - \frac{\nu^2}{4\pi} - \frac{\nu}{4} - \frac{1}{3\pi} + \mathcal{O}(\nu^{-1}) \right] + \mathcal{O}(s + 1).
\end{aligned} \tag{21}$$

In the following Section we will evaluate, as a function of  $\nu$ , the number of modes contributing in eq. (10), and in Section IV, their contributions to the vacuum energy.

### III. THE NUMBER OF CONTRIBUTING MODES

In this Section we address ourselves to the determination of  $\nu_0$  in (5), the maximum value of  $\nu$  for which  $N_\nu \geq 1$ . Although in the simple case under study the zeros of  $J_\nu(w)$  are well known [17], we prefer to establish a criterium which can be applied in more general situations.

First, notice that

$$N_\nu(x) \equiv \sum_{n=1}^{N_\nu} j_{\nu,n}^{-s} \Big|_{s=0} = [\zeta_\nu(s, 0^+) - \zeta_\nu(s, x)] \Big|_{s=0} \tag{22}$$

is a discontinuous function of  $x$ , having a step of height 1 at each positive zero  $j_{\nu,n}$  of the Bessel function  $J_\nu(w)$ .

Then,  $\nu_0$  can be determined from the condition

$$N_{\nu_0}(x) = N_{\nu_0}(j_{\nu_0,1} + 0) = 1, \tag{23}$$

with  $N_{\nu_0}(j_{\nu_0,1} - 0) = 0$ .

Taking into account eq. (19) and the fact that the second and third integrals in the r.h.s. of eq. (17) are real, it is straightforward to obtain from eqs. (15-17) that

$$\zeta_\nu(s = 0, x) = -\frac{\nu}{2} - \frac{1}{4} +$$

$$\begin{aligned}
&+ \Re \left\{ -\frac{i\nu}{\pi} \left( \sqrt{1 + e^{-i\pi} z^2} - \log(1 + \sqrt{1 + e^{-i\pi} z^2}) \right) \right. \\
&\quad \left. + \frac{i}{4\pi} \log(1 + e^{-i\pi} z^2) + \right. \\
&\quad \left. + \frac{i(2 + 3z^2)}{24\nu\pi(1 + e^{-i\pi} z^2)^{3/2}} \right\} + \mathcal{O}(\nu^{-2}),
\end{aligned} \tag{24}$$

where we have taken  $z = x/\nu \approx 1$ . In particular, for  $x \rightarrow 0^+$ ,

$$\zeta_\nu(s = 0, x = 0^+) = -\frac{\nu}{2} - \frac{1}{4} + \mathcal{O}(\nu^{-2}), \tag{25}$$

in coincidence with [18].

Now, taking the difference between (25) and (24) we get a *smooth* approximation,  $\tilde{N}_\nu(x) + \mathcal{O}(\nu^{-1})$ , to the step function  $N_\nu(x)$  in (22). It is easily seen that, for  $\nu > x$ ,  $\tilde{N}_\nu(x) = 0$  while, for  $\nu < x$ , we have

$$\begin{aligned}
\tilde{N}_\nu(x) &= \frac{\nu}{\pi} \left( \sqrt{z^2 - 1} - \arctan(\sqrt{z^2 - 1}) \right) - \\
&\quad - \frac{1}{4} - \frac{2 + 3z^2}{24\nu\pi(z^2 - 1)^{3/2}},
\end{aligned} \tag{26}$$

with  $z = x/\nu$ .

Let us now determine the value  $\nu_0$  for which

$$\tilde{N}_{\nu_0}(x) = 1/2. \tag{27}$$

To this end, we propose an expansion of the form

$$\sqrt{z_0^2 - 1} = \varepsilon_1 \nu_0^{-1/3} + \varepsilon_3 \nu_0^{-3/3} + \mathcal{O}(\nu_0^{-5/3}), \tag{28}$$

which makes sense for  $\nu_0 \gg 1$  and  $z_0 = x/\nu_0 \approx 1$ . Replacing in (26) and imposing (27), the coefficients  $\varepsilon_k$  can be determined order by order in  $\nu_0^{-1/3}$ , to get

$$x = \nu_0 + 1.857 \nu_0^{1/3} + 1.034 \nu_0^{-1/3} + \mathcal{O}(\nu_0^{-1}) \tag{29}$$

or, inverting this development,

$$\nu_0 = x - 1.857 x^{1/3} + 0.1155 x^{-1/3} + \mathcal{O}(x^{-1}). \tag{30}$$

(Notice that  $\nu_0 < x$ .)

Equation (29) is in excellent agreement with the expression of the first non-vanishing zero of  $J_{\nu_0}(w)$ , for a large order  $\nu_0$  [17]:  $j_{\nu_0,1} = \nu_0 + 1.8557 \nu_0^{1/3} + 1.03315 \nu_0^{-1/3} + \mathcal{O}(\nu_0^{-1})$ .

#### IV. THE DOMINANT CONTRIBUTIONS TO THE VACUUM ENERGY

In this Section we evaluate the first contributions to the vacuum energy obtained from the Debye expansion employed in Section II. As we will see, this allows to determine the volume, surface,  $\dots$ , terms in the Casimir energy of the scalar field.

According to the results in the previous Section, we are interested in the Laurent expansion of  $\zeta_\nu(s, x > \nu)$  and  $\zeta_\nu(s, x = 0^+)$  around  $s = -1$ , given in eqs. (20) and (21). As already remarked, the singular parts cancel out in the difference on the r.h.s. of eq. (10) (see eq. (19)). For the difference of the finite parts we get

$$\begin{aligned} z F(x, \nu) &\equiv [\zeta_\nu(s, 0^+) - \zeta_\nu(s, x)]_{s=-1} = \\ &= \frac{\nu^2}{2\pi} \left( z \sqrt{-1+z^2} - \log(z + \sqrt{-1+z^2}) \right) - \\ &- \frac{\nu}{4} + \frac{3z - 8z^3}{24\pi(-1+z^2)^{\frac{3}{2}}} + \frac{1}{8\pi} \log(z + \sqrt{-1+z^2}) \\ &+ \mathcal{O}(\nu^{-1}). \end{aligned} \quad (31)$$

This is a good approximation as long as  $\nu \gg 1$  and  $z = x/\nu \gtrsim 1$ .

Our aim is now to evaluate the sum in eq. (5),

$$\begin{aligned} E(R = x c/\Omega) &= \\ &= \frac{\hbar \Omega}{x} \sum_{\nu=1/2}^{\nu_0} \nu [\zeta_\nu(-1, x) - \zeta_\nu(-1, 0^+)] = \\ &= \hbar \Omega \sum_{\nu=1/2}^{\nu_0} F(x, \nu), \end{aligned} \quad (32)$$

with  $\nu_0$  given in (30).

The function  $F(x, \nu)$  is non-negative and has a pronounced maximum at  $\nu \approx x/2$  (i.e.  $z \approx 2$ ). Thus, the use of the approximation in eq. (31) is consistent if  $x \gg 1$ .

From eq. (31), it is not difficult to see that the successive terms in the Euler - Maclaurin summation formula [17],

$$\begin{aligned} \sum_{\nu=1/2}^{\nu_k} F(x, \nu) &= \int_{1/2}^{\nu_0} F(x, \nu) d\nu + \\ &+ \frac{1}{2} \{F(x, \nu_0) + F(x, 1/2)\} + \\ &+ \frac{1}{12} \partial_\nu [F(x, \nu)] \Big|_{\nu=1/2}^{\nu=\nu_0} + \dots, \end{aligned} \quad (33)$$

are of increasing order in  $x^{-1}$ . So, retaining the first few terms consistent with the approximation made, we get for the vacuum energy

$$\begin{aligned} \frac{E(R)}{\hbar \Omega} &= \\ &x^3 \left( \frac{-(\sqrt{-1+z_0^2}(-5+2z_0^2))}{24\pi z_0^3} + \frac{2z_0^4-3\log(z_0+\sqrt{-1+z_0^2})}{24\pi z_0^4} \right) - \\ &- x^2 \left( \frac{-\sqrt{-1+z_0^2}}{4\pi z_0^2} + \frac{\pi+3\log(z_0+\sqrt{-1+z_0^2})}{12\pi z_0^3} \right) - \\ &- x \left( \frac{-(-27+17z_0^2)}{48\pi z_0 \sqrt{-1+z_0^2}} + \frac{6\pi-10z_0^2+69\log(z_0+\sqrt{-1+z_0^2})}{48\pi z_0^2} \right) \\ &+ \mathcal{O}(x^0), \end{aligned} \quad (34)$$

where  $z_0 = x/\nu_0$ ,  $x = R\Omega/c$  and  $\nu_0$  is given in eq. (30). In this equation one can recognize volume, surface and curvature contributions to  $E(R)$ . Notice that we could have retained any number of terms of the asymptotic expansion in eq. (13), and performed the same steps as in the present calculation, to get the Casimir energy to any order in  $x^{-1}$ .

Finally replacing in eq. (34)  $z_0 \rightarrow x/\nu_0$  and  $\nu_0$  by its expression we get

$$E(R) = \hbar \Omega \left[ \frac{x^3}{12\pi} - \frac{x^2}{12} - 0.1343 x^{\frac{4}{3}} + \mathcal{O}(x) \right], \quad (35)$$

where one can see that the volume and surface terms are dominant for  $x \gg 1$ . Notice that also non-integer powers of the radius  $R$  appear as a consequence of the relation between  $\nu_0$  and  $x$ , eq. (30).

#### V. CONCLUSIONS

In equations (34-35) we have derived the dominant contributions to the vacuum energy of a scalar field in a model with a frequency dependent boundary condition, consisting in the confinement of the modes with low frequency (up to a physical cut-off  $\Omega$ ) to the interior of a sphere of radius  $R$ .

These modes are subject to Dirichlet boundary conditions at the surface of the sphere, while those with frequency higher than  $\Omega$  are free, being the boundary completely transparent to them. This characteristic of the model allows for the subtraction of the contribution of the high frequency modes to the vacuum energy, which amounts (independently of the regularization employed to define it) to an  $R$ -independent redefinition of the zero energy level, having no consequences on the evaluation of energy differences.

In so doing, we have represented the sum over the eigenfrequencies up to the cut-off  $\Omega$  in terms of an *incomplete*  $\zeta$ -function associated with the Laplacian operator in the sphere with Dirichlet boundary conditions (see eq. (10)). The function  $\zeta(s, x)$  as in eq. (12) is well defined only for  $\Re(s) > 1$ . So, it was analytically continued from  $s > 1$  to the relevant values of this parameter ( $s = 0$ , needed to evaluate the maximum angular momentum  $l_0 = \nu_0 - 1/2$  giving rise to eigenfrequencies less than or equal to  $\Omega$ , and  $s = -1$ , necessary to evaluate the contribution to the Casimir energy of the modes with angular momentum  $l = \nu - 1/2$ ) by approximating the behavior of the integrand in eq. (12) employing the Debye asymptotic expansion of the modified Bessel functions appearing in its expression.

This procedure has lead to a meromorphic function having simple poles with (exactly evaluated) cut-off independent residues (see eq. (19)), a necessary condition to have a finite result for the sum in eq. (10).

The finite part of  $\zeta(s, x)$  has been evaluated up to terms of a given order in  $\nu^{-1}$ . Although we have retained only the first terms in this asymptotic expansion, one can follow the same steps to determine the finite part of  $\zeta(s, x)$  at  $s = -1$  or 0 up to any required precision in  $\nu^{-1}$ .

Finally, the application of the Euler - Maclaurin summation formula has lead to an expression for the Casimir energy of the model in which one can recognize volume, surface and curvature contributions (see eq. (34)).

For a cut-off corresponding to  $x = R\Omega/c \gg 1$ , the dominant terms in the vacuum energy, eq. (35), are proportional to the volume ( $V = 4\pi R^3/3$ ) and area ( $S = 4\pi R^2$ ) of the sphere,

$$\frac{E(R)}{\hbar\Omega} = V \frac{\Omega^3}{16\pi^2 c^3} + \xi S \frac{\Omega^2}{12\pi^2 c^2} + \dots, \quad (36)$$

with  $\xi = -\pi/4$ .

It is worthwhile to remark that, for a similar model where the low frequency modes of the scalar field are subject to Neumann (rather than Dirichlet) boundary condition, we get the same expression for the dominant terms with  $\xi = +\pi/4$ .

These two dominant terms are in complete agreement with those obtained from the expansion of the density of states in powers of the inverse wavelength<sup>2</sup> [19,11]. The

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<sup>2</sup>Indeed, for scalar fields subject to local homogeneous boundary conditions the density of states is modified by finite volume effects [19]. The first correction in the asymptotic expansion for large wavelength  $k$  is given by

$$\sum_n \sim V \int \frac{d^3k}{(2\pi)^3} + S \int \xi \frac{d^3k}{(2\pi)^3 k} + \dots, \quad (37)$$

where the coefficient  $\xi$  takes the value  $\xi = -\pi/4$ ,  $\xi = +\pi/4$  and  $\xi = +\pi/4$  for Dirichlet, Neumann and Robin boundary

relation between  $\nu_0$  and  $x$ , eq. (30) (or, equivalently, the expression of the first zero of the Bessel function  $J_{\nu_0}(w)$  in terms of the order  $\nu_0$  [17]) introduces also non-integer powers of the radius  $R$  (see eq. (35)).

As a final exercise, we can use eq. (35) in a very schematic model pretending to mimic the phenomenon of sonoluminescence. We will adopt the values of the radius and emitted energy corresponding to a typical sonoluminescent bubble [4], and estimate the cut-off  $\Omega$  needed to produce this amount of energy. To this end, we will simply take the difference of the low frequency contribution to the Casimir energy of the scalar field for two different values of the bubble radius.

If the bubble collapses from an initial radius  $R = 4 \times 10^{-5} m$  to a final radius of one tenth this value, and the emitted energy is  $\mathcal{E} = 1.2 \times 10^{-12} \text{ Joule}$ , by imposing the equality

$$\frac{R}{\hbar c} (E(R) - E(R/10)) = 1.516 \times 10^9, \quad (38)$$

and taking into account eq. (35), it follows that  $x = 490$ , justifying the use of the approximation obtained.

This implies that  $\Omega = 3.675 \times 10^{15} \text{ 1/sec}$ , which corresponds to a cut-off in wavelengths in the ultraviolet of  $\Lambda = 5.129 \times 10^{-7} m = 5129 \text{ \AA}$ , not far from the region where the refraction index of water becomes essentially 1 [20]. This strongly suggests to consider a similar model for the case of the electromagnetic field in the presence of dielectric media, calculation which will be presented elsewhere [15].

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[1] H.B.G. Casimir. *Proc. Kon. Nederl. Akad. Wet.*, 51:793-796, 1948.

[2] V. M. Mostepanenko and N. N. Trunov. *The Casimir effect and its applications*. Oxford Science Publications, New York, 1997.

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conditions respectively. Then, introducing the dispersion relation  $\omega(k) = ck$  and a cut-off in the wavelength given by  $K = \Omega/c$ , it is easy to get eq. (36) for the vacuum energy.

For *junction* boundary conditions, as those appearing in problems with dielectrics, the coefficient  $\xi$  has more involved expressions, which can be difficult to evaluate [11].

- [3] J. Schwinger. *Proc. Nat. Acad. Sci.*, 89, (1992) 4091; 89, (1992) 11118; 90, (1993) 958; 90, (1993) 2105; 90, (1993) 4505; 90, (1993) 7285; 91, (1994) 6473.
- [4] R. Löfstedt B.P. Barber, R.A. Hiller and S.J. Putterman. *Physics Reports*, 281:65–143, 1997.
- [5] Kimball A. Milton and Y. Jack Ng. Casimir energy for a spherical cavity in a dielectric: Applications to sonoluminescence. hep-th/9607186, 1996.
- [6] Kimball A. Milton and Y. Jack Ng. Observability of the bulk casimir effect: Can the dynamical casimir effect be relevant to sonoluminescence? *Phys. Rev.*, E57:5504, 1998.
- [7] Iver Brevik, V. N. Marachevsky, and Kimball A. Milton. Identity of the van der waals force and the casimir effect and the irrelevance of these phenomena to sonoluminescence. *Phys. Rev. Lett.*, 82:3948–3951, 1999.
- [8] J. S. Hoyer and I. Brevik. The casimir problem of spherical dielectrics: Quantum statistical and field theoretical approaches. quant-ph/0008088, 2000.
- [9] M. Bordag, K. Kirsten, and D. Vassilevich. On the ground state energy for a penetrable sphere and for a dielectric ball. *Phys. Rev.*, D59:085011, 1999.
- [10] G. Lambiase, G. Scarpetta, and V. V. Nesterenko. Exact value of the vacuum electromagnetic energy of a dilute dielectric ball in the mode summation method. hep-th/9912176, 1999.
- [11] C. Molina-Paris and Matt Visser. Casimir effect in dielectrics: Surface area contribution. *Phys. Rev.*, D56:6629–6639, 1997.
- [12] C. E. Carlson, C. Molina-Paris, J. Perez-Mercader, and Matt Visser. Casimir effect in dielectrics: Bulk energy contribution. *Phys. Rev.*, D56:1262–1280, 1997.
- [13] S. Liberati, Matt Visser, F. Belgiorno, and D. W. Sciama. Sonoluminescence as a qed vacuum effect. ii: Finite volume effects. *Phys. Rev.*, D61:085024, 2000.
- [14] S. Liberati, Matt Visser, F. Belgiorno, and D. W. Sciama. Sonoluminescence as a qed vacuum effect. i: The physical scenario. *Phys. Rev.*, D61:085023, 2000.
- [15] H. Falomir and K. Rébora. to be published. 2000.
- [16] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products*. Academic Press, Inc., New York, 1980.
- [17] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, Inc., New York, 1972.
- [18] A. Romeo. Bessel  $\zeta$ -function approach to the Casimir effect of a scalar field in a spherical bag. *Physical Review D*, 52:7308–7314, 1995.
- [19] R. Balian and C. Bloch. Distribution of eigenfrequencies for the wave equation in a finite domain: Three-dimensional problem with smooth boundary surface *Ann. Phys.*, 60:401–447, 1970.
- [20] J. D. Jackson. *Classical Electrodynamics*, 2nd. ed. Wiley, New York, 1975.